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Do Diophantine vectors form a Cantor bouquet?

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Dedicated to Robert Devaney on the occasion of his 60th birthday

We introduce the set of Diophantine vectors in \mathbb{R}^n , which is a standard notion in KAM Theory. The problem is whether this set forms a Cantor bouquet.

Keywords: Diophantine vector; resonant hyperplane; Cantor set; Cantor bouquet

AMS Classification: 11J25; 11J70; 54E52; 54G05

1. Definition and elementary properties

Let $n \geq 2$ and for $\tau > 0$ and $\gamma > 0$ consider the set of (τ, γ) -Diophantine frequency vectors by

$$\mathbb{R}_{\tau, \gamma}^n = \{\omega \in \mathbb{R}^n \mid \forall k \in \mathbb{Z}^n \setminus \{0\} : |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}\}.$$

Here, $\langle \omega, k \rangle = \sum_{j=1}^n \omega_j k_j$ is the usual inner product and $|k| = \sum_{j=1}^n |k_j|$ is the length of the integer vector k .

We summarize a few properties of $\mathbb{R}_{\tau, \gamma}^n$ (for details see [3,4]). First of all, the set $\mathbb{R}_{\tau, \gamma}^n$ is a union of closed half lines in the sense that for $\omega \in \mathbb{R}_{\tau, \gamma}^n$, then $s\omega \in \mathbb{R}_{\tau, \gamma}^n$ for all $s \geq 1$. Moreover, if \mathbb{S}^{n-1} denotes the unit sphere in \mathbb{R}^n , we consider the intersection $\mathbb{R}_{\tau, \gamma}^n \cap \mathbb{S}^{n-1}$. This is a closed (and hence compact) set that by the Cantor–Bendixson Theorem is the union of a perfect and a discrete set. Due to the fact that for any $k \in \mathbb{Z}$ the resonant hyperplane

$$\{\omega \in \mathbb{R}^n \mid \langle \omega, k \rangle = 0\},$$

is in its complement, the perfect set is totally disconnected and hence is a Cantor set.

For $\tau > n - 1$, the intersection $\mathbb{R}_{\tau, \gamma}^n \cap \mathbb{S}^{n-1}$ has positive measure in \mathbb{S}^{n-1} for sufficiently small $\gamma > 0$. Indeed, one even has that the measure of the complement $\mathbb{S}^{n-1} \setminus \mathbb{R}_{\tau, \gamma}^n$ is of order $O(\gamma)$ as $\gamma \downarrow 0$. For a sketch of the planar case $\mathbb{R}_{\tau, \gamma}^2 \subset \mathbb{R}^2$, see Figure 1.

For $n = 2$, by taking the ratio $\varrho = \omega_1/\omega_2$, one finds a relationship with the Diophantine numbers $\varrho \in \mathbb{R}$ satisfying

$$\left| \varrho - \frac{p}{q} \right| \geq \gamma q^{-(\tau+1)},$$

for all rationals $p/q \in \mathbb{Q}$, which is a subset of the Bruno numbers [7]. Compare with the horizontal line in Figure 1. Generally, for $n \geq 3$, a similar relationship is established

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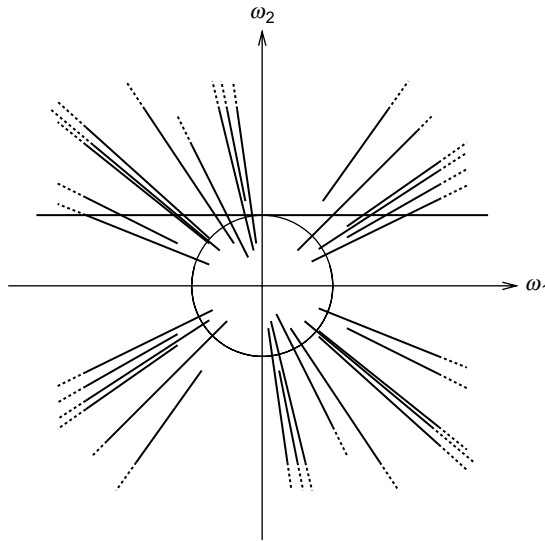


Figure 1. The Diophantine set $\mathbb{R}^n_{\tau, \gamma}$ has closed half line geometry and the intersection $\mathbb{S}^{n-1} \cap \mathbb{R}^n_{\tau, \gamma}$ is a Cantor set of measure $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau, \gamma} = O(\gamma)$ as $\gamma \downarrow 0$.

between $\omega \in \mathbb{R}^n$ and $[\omega_1 : \omega_2, \dots, \omega_n] \in \mathbb{P}^{n-1}(\mathbb{R})$, the $(n-1)$ -dimensional projective space [2].

2. The problem

The problem is to show that $\mathbb{R}^n_{\tau, \gamma}$ is a Cantor bouquet [1,5]. For this, it is needed to know more about the endpoints of the closed half lines in $\mathbb{R}^n_{\tau, \gamma}$, in particular, one has to prove that the set of endpoints accumulates on every point of $\mathbb{R}^n_{\tau, \gamma}$.

For $n = 2$, the answer may be related to a similar result that concerns the graph of the Bruno numbers, which uses continued fraction expansions [6]. For $n \geq 3$, the number theoretic aspects of the sets $\mathbb{R}^n_{\tau, \gamma}$ and $\mathbb{R}^n_{\tau, \gamma} \cap \mathbb{S}^{n-1}$ may be a lot more involved.

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